3.6B Short Term Coupling Midway through a Trial

The independence result for short-term coupling is extended to interactions occurring partway through a trial

Version of 5/23/03

Introduction

In section 3.6 (all references are to *Bigger than Chaos*), I stated the following, rather strong result: the independence of outcomes produced by microconstant trials is not compromised by a short term coupling, provided that (a) the individual and joint IC-densities are macroperiodic, (b) the coupling can be represented by a non-deflationary microlinear transformation, and (c) the coupling occurs at the beginning of the trial. The result is strong because it applies to all microconstant experiments, regardless of their mechanics. But it is considerably less strong than might be desired, due to the requirement that the coupling occur at the very beginning of the trials. When I apply the result in the vindication of EPA (section 4.54), I individuate trials so that this requirement is satisfied. But although the issue is thereby rendered unimportant for the purposes of explaining the simple behavior of complex systems, it is interesting to ask: under what conditions can the requirement be relaxed? How can the result be generalized so as to apply to couplings occurring in mid-trial?

One benefit of such a generalization would be the ability to understand the independence of enion probabilities without analyzing them in terms of microdynamic probabilities.[1] The various interactions between enions that appear to compromise independence would be shown to be the sort of short term coupling that does not affect independence.[2]

It is not hard to see that a complete generalization of the timing of the short term coupling is impossible. The reason is that a particular experiment's microconstancy might be due to an elaborate mechanism whose operation depends very sensitively on some intermediate state. If a coupling upsets that state, the evolution function for the experiment might

1. The microdynamic analysis would still be needed, however, to show that enion probabilities are microconstant.

2. This approach is taken in my paper "How are the sciences of complex systems possible", available on the *Bigger than Chaos* web site.

be as different from the evolution function for its uncoupled counterpart as you like. A result that allows mid-trial couplings must, unlike the result stated in section 3.6, put some restrictions on the kind of mechanism on which the trials occur.

In what follows, I do two things. First, I show that for the case of the colliding coins discussed in section 3.6, the generalization to mid-trial couplings is straightforward. Second, I make some comments about a much more extensive generalization of the result. Although this more extensive generalization can be justified without too much difficulty, it requires quite a bit of setting up and the generalization itself is rather messy. It is merely sketched here.

Colliding Coins

In this section I show that, if the two tossed coins of section 3.6 collide at any point in their flight, their outcomes are independent provided that the collision effects a microlinear, non-deflationary transformation of the coins' spin speeds. What I prove below is actually that the effect of a microlinear non-deflationary transformation partway through the coins' flight is equivalent to a different microlinear non-deflationary transformation applied at the beginning of the flight. It follows, from the result stated in section 3.6, that the outcomes of the tosses are independent.

Suppose, then, that two coins are tossed, with spin speeds of ζ_1 and ζ_2 respectively, in such a way that they both spin for time t. At some time t_α less than t, the coins collide. The state of each coin at the time of the collision can be represented by the number of revolutions that it has made since it was launched. It what follows, I will focus on the dynamics of the first coin. Let n_α, then, be the number of revolutions made by the first coin at the time of the collision t_α. Supposing that spin speeds are represented in revolutions per time unit,

$$n_\alpha = t_\alpha \zeta_1.$$

If the speed of the coin is altered at t_α to ζ'_1, then the number of revolutions the coin makes between t_α and t is

$$t_\beta \zeta'_1$$

where $t_\beta = t - t_\alpha$. Thus the total number of revolutions n made by the coin is

$$\begin{aligned} n &= t_\beta \zeta_1' + n_\alpha \\ &= t_\beta \zeta_1' + t_\alpha \zeta_1. \end{aligned}$$

I will first consider the case where the transformation effected by the coupling is linear, and so can be represented as follows:

$$\zeta_1' = a\zeta_1 + b\zeta_2 + c$$

for some a, b, and c. The total number of revolutions made by the coin is the sum of the revolutions made before the collision and the revolutions made after the collision:

$$\begin{aligned} n &= t_\beta \zeta_1' + t_\alpha \zeta_1 \\ &= t_\beta (a\zeta_1 + b\zeta_2 + c) + t_\alpha \zeta_1 \\ &= (at_\beta + t_\alpha)\zeta_1 + (bt_\beta)\zeta_2 + ct_\beta. \end{aligned}$$

My aim is to find a linear or linear plus constant transformation of ζ_1 and ζ_2 that yields the same formula for n when applied to ζ_1 at the very beginning of the toss. This new transformation, if it exists, has the form:

$$d\zeta_1 + e\zeta_2 + f.$$

If a transformation of this form is applied at the beginning of a toss, the total number of revolutions made by the coin is:

$$\begin{aligned} n &= t(d\zeta_1 + e\zeta_2 + f) \\ &= dt\zeta_1 + et\zeta_2 + ft. \end{aligned}$$

Such a transformation is equivalent to the actual transformation if there are values of d, e, and f for which these two formulae for n are equivalent. Clearly, there are:

$$\begin{aligned} d &= (at_\beta + t_\alpha)/t \\ e &= bt_\beta/t \\ f &= ct_\beta/t. \end{aligned}$$

Define $\hat{t}_\alpha$ and $\hat{t}_\beta$ as the proportions of the total spin time that elapse before and after the collision, respectively, that is

$$\hat{t}_\alpha = t_\alpha/t$$
$$\hat{t}_\beta = t_\beta/t.$$

Then d, e, and f can be written

$$d = a\hat{t}_\beta + \hat{t}_\alpha$$
$$e = b\hat{t}_\beta$$
$$f = c\hat{t}_\beta.$$

So the effect of a linear coupling partway through a coin toss is equivalent to the effect of a linear coupling—though a different one—at the beginning of the toss.

Under what conditions will the equivalent coupling at the beginning of the toss be non-deflationary? The linear plus constant transformation of ζ_2 within any region may be represented as follows:

$$\zeta_2' = g\zeta_2 + h\zeta_1 + i.$$

Then, as shown in the appendix at the end of this section, a sufficient condition for the coupling at the beginning of the toss to be non-deflationary is that:

1. The coupling partway through the toss is non-deflationary (or equivalently, $ag - bh \geq 1$), and
2. $a + g \geq 2$.

If it is assumed that a and g are positive and that (because of physical symmetry) the signs of b and h are the same, then (1) entails that $ag \geq 1$, which in turn entails (2). Under these assumptions, then, if the coupling transformation partway through the toss is non-deflationary, its equivalent at the beginning of the toss is also non-deflationary.[3]

3. Why, if a and g are not positive, is a non-deflationary transformation partway through a toss not equivalent to a non-deflationary transformation at the beginning of the toss? When, say, a is negative, the result of a collision is that the coin starts spinning in reverse. This creates the potential for the time evolution of the coin after

Now suppose that the transformation effected by the coupling is microlinear, that is, linear over any micro-sized, contiguous set of values of ζ_1 and ζ_2. From the reasoning above, over any such region R, the effect of the transformation for initial spin speeds falling into R is equivalent to a different linear coupling at the beginning of the trial. Thus the effect of a microlinear coupling partway through a coin toss, which is made up of different linear couplings over different micro-sized regions, is approximately equivalent to the effect of a corresponding set of linear couplings over the same regions at the beginning of a coin toss, thus is equivalent to a microlinear coupling at the beginning of the toss. Provided that the original coupling is non-deflationary, the new coupling is also non-deflationary.

The same sort of demonstration can be carried out for various more complicated experiments. For example, the demonstration is fairly simple for the case of two colliding coins that each take two IC-variables, spin speed and spin time, even though in this case, the number of revolutions completed is not a linear function of the IC-variables. (Coin tosses with two IC-variables are discussed in section 2.25.)

A More General Result for Stirring Experiments

It would be highly desirable to generalize this result far beyond the case of the coin. Such a generalization is quite possible. It can be shown that the independence of a large and important class of microconstant experiments, which I call the *stirring* experiments, is usually unaffected by a short term coupling acting at any time that is microlinear and non-deflationary. In what follows I provide a brief overview of the independence result for stirring experiments.

The Stirring Experiments The stirring experiments include the tossed coin and many other mechanical gambling devices, as well as perhaps the systems that make up the subject matter of statistical physics (see section 4.8). Their stirring aspect can be quite complex; the following characterization of stirring ignores most of these complexities. It is still rather

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the collision to undo the effect of the time evolution before the collision. The net effect may be as if the coin only makes a few revolutions, perhaps too few for microconstancy. The conditions under which such a collision will not destroy the microconstancy of the composite evolution function, then, are rather more complicated than in the case where a and g are positive, that is, where the coins continue spinning in the same direction after the collision.

long, but the pay-off for a long setup will be a relatively brief treatment of coupling itself.

The simplest stirring experiments have an internal variable that I call the experiment's *determiner* (but not all experiments with a determiner are stirring). In this subsection, I first say what it is for an internal variable to be a determiner, and then state the additional properties that a determiner must have if an experiment is to count as stirring.

An internal variable of an experiment is a determiner if it has the following properties:

1. At the beginning of a trial, it has either a single given value or a value that is fixed by the experiment's IC-variables.

2. Over the course of the trial, the value of the determiner changes in a way that is dictated by the experiment's IC-variables.

3. At the end of a trial, the value of the determiner determines the trial's outcome.

A determiner of the tossed coin experiment, for example, is the total number of revolutions described by the coin.[4] This number starts out at zero, grows at a constant rate determined by the spin speed of the coin, and at the end of the toss, determines whether the outcome is heads or tails.

An experiment is stirring if its determiner has the following properties. First, as the experiment goes on, the range of possible values for the determiner gets larger and larger; that is, the space of possible values is inflated. Second, this inflation is reasonably smooth; specifically, it must have a certain sort of microlinearity. Third, the function that takes as its argument the final value of the determiner and yields the outcome of the experiment must have a kind of microconstant aspect, except that the constant ratio partition for the function need not be micro-sized. Under these conditions, the experiment will be microconstant; stirring, then, causes microconstancy.

4. Another obvious choice of determiner is the fractional part of the total number of revolutions, since this alone is enough to determine the outcome of the kind of simple coin toss characterized in section 2.25. But it will soon be apparent that the tossed coin experiment is not *stirring* with respect to this choice of determiner.

The Coin as Stirrer Because this is a both a rather abstract and rather vague list of conditions, let me show how in a concrete and specific case, that of the tossed coin, the satisfaction of these conditions leads to microconstancy. Consider the conditions in reverse, beginning with the function that maps the final value of the tossed coin's determiner—the total number of revolutions the coin has completed at the end of the toss—to the outcome of the toss. Like an evolution function, this function, which I call the *indicator function* for the toss, maps to zero or one, depending on whether the designated outcome occurs. In the case of the coin (as described in section 2.25), the indicator function for heads maps any number of revolutions with a fractional part between zero and one half to zero, meaning tails, and any number of revolutions with a fractional part greater than one half to one, meaning heads. The indicator function, then, has an optimal constant ratio partition consisting of the intervals $[n, n+1]$ (n an integer) where n is the final number of revolutions completed. Each member of this partition has strike ratio 1/2 for heads. The function, then, has a CRI of 1. (Because the indicator function is not an evolution function, this is strictly speaking an extension of the notion of an optimal constant ratio partition.)

Now consider the mapping that takes a particular value of the tossed coin's only IC-variable, its spin speed ζ, to a particular final value for the determiner. Call this mapping the determiner evolution function $d(\zeta)$. The determiner evolution function, the indicator function $I_e(\cdot)$, and the evolution function for an outcome are related as follows:

$$h_e(\zeta) = I_e(d(\zeta)).$$

Suppose that V is member of an optimal constant ratio partition for the indicator function. (V, then, is a set of contiguous values of the determiner.) Consider the inverse image of V under the determiner evolution function. The inverse image is a set of values of ζ; call it U. A sufficient condition for the microconstancy of the evolution function is that each such U be:

1. Contiguous,
2. Micro-sized, and
3. Have the same strike ratio for heads as V.

Then the inverse image of the optimal constant ratio partition for the indicator function—which is just the collection of all the Us—will be a micro-sized constant ratio partition for the experiment.

In the case of the tossed coin, all three conditions are satisfied. The determiner evolution function for the coin is:

$$d(\zeta) = t\zeta$$

where t is the fixed spin time of a toss. The function is linear with determinant t. Because of linearity, conditions (1) and (3) are satisfied, for reasons discussed in section 3.64. Provided that t is large enough, condition (2) is also satisfied.

How large is large enough depends on (a) the CRI of the indicator function's constant ratio partition, and (b) the smoothness of the relevant IC-density. If the CRI of the partition is a, the CRI of the partition's inverse image is a/t, so the larger t is, the smaller the evolution function's CRI (the evolution function's CRI being the CRI of its optimal constant ratio partition, hence of the inverse image), and so the more likely that the IC-density will be macroperiodic relative to this partition. Since the CRI for the coin's indicator function is 1, the CRI of the inverse image is $1/t$. Inflation is sufficient, then, if the coin's IC-density is macroperiodic relative to a partition of this size, that is, if the IC-density is approximately uniform over any interval of width $1/t$.

Using the tossed coin as a paradigm, it should now be easy to see that any given stirring experiment will likely be microconstant. The reason is that the inverse image (under the determiner evolution function) of a stirring experiment's indicator function's optimal constant ratio partition will likely be a micro-sized constant ratio partition for the same outcome.

It will likely be a constant ratio partition due to the microlinearity of the determiner evolution function. In the case of the tossed coin, the constant strike ratio was due to the determiner evolution function's linearity, but because the determiner evolution function need only be linear over one member of the indicator function's constant ratio partition—over one V—at a time, microlinearity is sufficient.

The constant ratio partition will likely be micro-sized because the determiner evolution function is inflationary. More specifically, if the CRI of an optimal constant ratio partition for the indicator function is a, and the largest partition of initial conditions that would count as micro-sized has CRI b, then a microlinear determiner evolution function is sufficiently inflationary for stirring if each of its linear parts has a determinant of at least a/b.

In order to obtain the independence result, the conditions for stirring must be understood in a certain way. They must be understood to imply that the evolution of the determiner is microlinear at all times, by which I mean that for any two times t_1 and t_2, the evolution of the determiner between t_1 and t_2 is a microlinear function of the initial conditions and the value of the determiner at t_1.

This rules out the possibility that the microlinearity of determiner evolution is the result of a series of non-microlinear transformations in which the non-linearities just happen to cancel out so as to create a microlinear transformation. Obviously, the coin satisfies this constraint, since the number of revolutions made between t_1 and t_2 is given by a linear function, $\zeta(t_2 - t_1)$, and so the total number of spins made at t_2, given that n spins have been made by t_1, is $n + \zeta(t_2 - t_1)$, also linear. The constraint is important in the treatment of coupling because I will assume that the evolution of the determiner both before and after the moment of coupling is microlinear.

Note that the time evolution of the determiner in a stirring experiment has in a certain sense the stretch-and-fold aspect that accounts for the sensitivity to initial conditions of chaotic systems. The stretching is performed by the inflationary determiner evolution function; the folding by the indicator function (the folds themselves being located at the borders of the members of the optimal constant ratio partition).

Short Term Couplings between Stirring Experiments A short term coupling will leave independence unaffected if it does not interfere with those properties of a stirring experiment in virtue of which it is microconstant.[5] Very roughly, then, the coupling must not affect either the microlinearity or the inflationary power of the determiner evolution function. I claim that, provided the coupling is itself microlinear and non-deflationary, independence will tend to be maintained.

Let me sketch, briefly, the reasons for this tendency, noting at the same time the various ways in which exceptions may arise. There are three transformations to take into account:

5. Strictly speaking, it is the stirring aspect of the *composite* evolution function that concerns us here.

1. The first stage of the determiner's evolution, before the coupling,
2. The coupling itself, and
3. The second stage of the determiner's evolution, after the coupling.

What needs to be shown is that the composition of all three stages is a microlinear, sufficiently inflationary mapping from the initial condition space to the final state of the determiner.

It is important to note (a) that the second stage of the transformation is rather different from first and third stages, and (b) that the third stage is subtly different from the first stage. Let me elaborate.

First, the difference between the second stage and the others. This difference is readily apparent in an experiment like the double coin toss: whereas, in the first and third stages, the determiner changes but the initial conditions—the spin speeds of the coins—remain the same, in the second stage, the determiner does not change but the spin speeds do. The best way to conceptualize the process is as a series of three operations on a subset of the Cartesian product of the determiner space and the initial condition space.

Second, the subtle difference between the first and third stages. This difference is due to the fact that, while the first stage is a function of the initial conditions only, the third stage is a function of the value of the determiner and the (transformed) initial conditions. Take the case of the tossed coin, for example. Divide the total time that the coin is spinning into two periods of length t_α and t_β, as above. Then if ζ is the initial spin speed and ζ' is the spin speed transformed by a coupling after t_α, the first and third stages of determiner evolution may be represented by the following two linear transformations respectively: $f(\zeta) = t_\alpha \zeta$ and $g(\zeta', n) = n + t_\beta \zeta'$ (where $g(\cdot)$'s argument n is the number of spins completed after t_α, that is, $f(\zeta)$).

Now I will explain why a non-deflationary, microlinear coupling will likely not destroy the independence of two stirring experiments. First, a piece of terminology: since the initial conditions in the coupled experiments undergo a change, it is quite forced to continue to refer to them as such; call them instead the evolution parameters, since they drive the time evolution of the determiner.

It is easy to see that adding a non-deflationary transformation between the first and third stages will likely not diminish the inflationary power of the whole. Indeed, an inflationary second stage will tend to increase the inflationary power of the whole, because it increases the range of evolution parameters acting on a given area of determiner space, so extending the range of operations acting on the area and therefore likely extending the range of possible end-points of the operations.[6]

There is, however, a large class of exceptions to this observation: the effect of the second stage may be to cause the evolution parameters to take on new values that cause the evolution of the determiner in the third stage to be less inflationary than it is without the coupling, or even deflationary. The case where a collision reverses the direction of the coins' spins is an example. What is required to maintain inflation is something like this: the new values of the evolution parameters must be inflationary *in the same directions* as the old, which in the simple one-determiner case means having a determinant of the same sign (and for inflation, of magnitude greater than one).

Next, microlinearity. A general tendency is clear: inserting a linear transformation between two other linear transformations will not compromise the linearity of the whole. But again there are pitfalls.

The first sort of exception is a little like the exception in the case of inflation: the coupling may transform the parameters so they take on values (presumably not possible as initial conditions) for which determiner evolution is not microlinear.

The second sort of exception arises from the following insidious fact: when the coupling is inflationary, the microlinearity of the whole will tend

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6. I should note that it is possible for an inflationary linear transformation on a j-dimensional space such as $n \times \zeta$ to *deflate* a subset of dimension less than j. To put the point informally, you can make a region bigger as a whole while reducing its width in some particular direction (the reduction being made up for by increases in other directions).

Since we ultimately care only about the inflation of the determiner, hence only about the change in the size of the projection of the evolution parameter/determiner space (for the coin, $n \times \zeta$) onto the determiner space, this is a potential pitfall for claims like the one made above. In practice, we are fine provided that the transformations that make up the complete determiner evolution function each act exclusively on either the evolution parameters or the determiner. Then, if they are inflationary, they are inflationary in virtue of their effect on the influenced variable(s), hence they inflate the projection onto that variable's (or those variables') subspace, leaving the other variables untouched. The subspace of interest—the determiner space—is therefore always either inflated or unaffected by an inflationary transformation.

to be diminished. To see this, partition the space of the evolution parameters driving the third stage into contiguous regions that are as large as possible but within each of which the third stage acts as an approximately linear transformation. This partition $\mathcal{P}$ in effect measures the microlinearity of the third stage. The composition of all three stages is microlinear in part because points that are close together at the beginning of the process are still close enough at the end of the second stage that they belong to the same member of $\mathcal{P}$. An inflationary second stage will tend to spread out the points in parameter space, and so make it less likely that such points fall into the same member of $\mathcal{P}$.

Yet there is compensation for this. As noted above, the more inflationary the second stage, the more inflationary the complete transformation is likely to be. But the more inflationary the transformation, the less microlinear it needs to be for microconstancy to hold.[7] Thus an inflationary second stage works both for and against microconstancy, hence both for and against independence. Insofar as these two effects tend to cancel out, inflationary couplings will pose no net danger.

Let me summarize, this time explicitly mentioning the exceptions. In an experiment that is microconstant due to stirring, a non-deflationary, microlinear coupling will tend not to destroy independence, provided that (a) all possible values of the determiner evolution parameters cause non-deflationary evolution *in the same directions* (b) all possible values of the determiner evolution parameters cause microlinear evolution, and (c) if the coupling is inflationary, the resulting increased inflation of the whole compensates for the reduced microlinearity of the whole. A more formal result will, I think, have to make much more specific assumptions about the form of determiner evolution.

7. Because microlinearity is required relative to the inverse image of the indicator function, and the size of this inverse image shrinks as determiner evolution becomes more inflationary.

Appendix: Sufficient Conditions for Non-Deflation

This appendix justifies the claim, used in the treatment of the colliding coins above, that a sufficient condition for a coupling at the beginning of the toss to be non-deflationary is that:

1. The equivalent coupling partway through the toss is non-deflationary, and
2. $a + g \geq 2$.

(See the earlier discussion for the meaning of these terms and parameters.)

The coupling transformation applied at the beginning of the toss is non-deflationary just in case its determinant is greater than or equal to 1. The determinant of the transformation is

$$\det \begin{pmatrix} a\hat{t}_\beta + \hat{t}_\alpha & b\hat{t}_\beta \\ h\hat{t}_\beta & g\hat{t}_\beta + \hat{t}_\alpha \end{pmatrix} = ag(\hat{t}_\beta)^2 + a\hat{t}_\alpha\hat{t}_\beta + g\hat{t}_\alpha\hat{t}_\beta + (\hat{t}_\alpha)^2 - bh(\hat{t}_\beta)^2$$

$$= (ag - bh)(\hat{t}_\beta)^2 + \frac{(a+g)}{2} 2\hat{t}_\alpha\hat{t}_\beta + (\hat{t}_\alpha)^2.$$

Since $(\hat{t}_\beta)^2 + 2\hat{t}_\alpha\hat{t}_\beta + (\hat{t}_\alpha)^2 = (\hat{t}_\alpha + \hat{t}_\beta)^2 = 1$, a sufficient condition for the determinant to be greater than or equal than 1 is that the coefficients of $(\hat{t}_\beta)^2$ and $2\hat{t}_\alpha\hat{t}_\beta$ in the above expression be greater than or equal to 1, that is, that

1. $ag - bh \geq 1$, and
2. $(a + g)/2 \geq 1$.

The quantity $ag - bh$ is just the determinant of the original transformation, that is, the transformation applied partway through the toss; it is greater than or equal to 1 just in case the original transformation is non-deflationary.